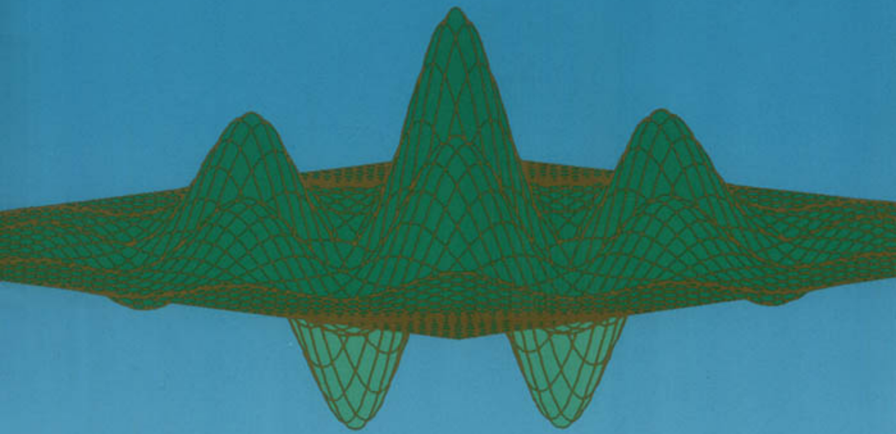
WAVELET ANALYSIS AND ITS APPLICATIONS VOLUME 1

An Introduction to WAVELETS



CHARLES K. CHUI

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An Introduction to
Wavelets

Wavelet Analysis and Its Applications

The subject of wavelet analysis has recently drawn a great deal of attention from mathematical scientists in various disciplines. It is creating a common link between mathematicians, physicists, and electrical engineers. This book series will consist of both monographs and edited volumes on the theory and applications of this rapidly developing subject. Its objective is to meet the needs of academic, industrial, and governmental researchers, as well as to provide instructional material for teaching at both the undergraduate and graduate levels.

This first volume is intended to be introductory in nature. It is suitable as a textbook for a beginning course on wavelet analysis and is directed towards both mathematicians and engineers who wish to learn about the subject. Specialists may use this volume as supplementary reading to the vast literature that has already emerged in this field.

This is a volume in
WAVELET ANALYSIS AND ITS APPLICATIONS

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A list of titles in this series appears at the end of this volume.

An Introduction to Wavelets

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To Margaret

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Preface

Fourier analysis is an established subject in the core of pure and applied mathematical analysis. Not only are the techniques in this subject of fundamental importance in all areas of science and technology, but both the integral Fourier transform and the Fourier series also have significant physical interpretations. In addition, the computational aspects of the Fourier series are especially attractive, mainly because of the orthogonality property of the series and of its simple expression in terms of only two functions: $\sin x$ and $\cos x$.

Recently, the subject of "wavelet analysis" has drawn much attention from both mathematicians and engineers alike. Analogous to Fourier analysis, there are also two important mathematical entities in wavelet analysis: the "integral wavelet transform" and the "wavelet series". The integral wavelet transform is defined to be the convolution with respect to the dilation of the reflection of some function $\tilde{\psi}$, called a "basic wavelet", while the wavelet series is expressed in terms of a single function ψ, called an "$\mathcal{R}$-wavelet" (or simply, a wavelet) by means of two very simple operations: binary dilations and integral translations. However, unlike Fourier analysis, the integral wavelet transform with a basic wavelet $\tilde{\psi}$ and the wavelet series in terms of a wavelet ψ are intimately related. In fact, if $\tilde{\psi}$ is chosen to be the "dual" of ψ, then the coefficients of the wavelet series of any square-integrable function f are precisely the values of the integral wavelet transform, evaluated at the dyadic positions in the corresponding binary dilated scale levels. Since the integral wavelet transform of f simultaneously localizes f and its Fourier transform $\hat{f}$ with the zoom-in and zoom-out capability, and since there are real-time algorithms for obtaining the coefficient sequences of the wavelet series, and for recovering f from these sequences, the list of applications of wavelet analysis seems to be endless. On the other hand, polynomial spline functions are among the simplest functions for both computational and implementational purposes. Hence, they are most attractive for analyzing and constructing wavelets.

This is an introductory treatise on wavelet analysis with an emphasis on spline-wavelets and time-frequency analysis. A brief overview of this subject, including classification of wavelets, the integral wavelet transform for time-frequency analysis, multiresolution analysis highlighting the important properties of splines, and wavelet algorithms for decomposition and reconstruction of functions, will be presented in the first chapter. The objective of this chapter is not to go into any depth but only to convey a general impression of what wavelet analysis is about and what this book aims to cover.

This monograph is intended to be self-contained. The only prerequisite is a basic knowledge of function theory and real analysis. For this reason, preliminary material on Fourier analysis and signal theory is covered in Chapters 2 and 3, and an introductory study of cardinal splines is included in Chapter 4. It must be pointed out, however, that Chapters 3 and 4 also contribute as an integral part of wavelet analysis. In particular, in Chapter 3, the notion of "frames", and more generally "dyadic wavelets", is introduced in the discussion of reconstruction of functions from partial information of their integral wavelet transforms in time-frequency analysis.

The common theme of the last three chapters is "wavelet series". Hence, a general approach to the analysis and construction of scaling functions and wavelets is discussed in Chapter 5. Spline-wavelets, which are the simplest examples, are studied in Chapter 6. The final chapter is devoted to an investigation of orthogonal wavelets and wavelet packets.

The writing of this monograph was greatly influenced by the pioneering work of A. Cohen, R. Coifman, I. Daubechies, S. Mallat, and Y. Meyer, as well as my joint research with X. L. Shi and J. Z. Wang. In learning this fascinating subject, I have benefited from conversations and correspondence with many colleagues, to whom I am very grateful. In particular, I would like to mention P. Auscher, G. Battle, A. K. Chan, A. Cohen, I. Daubechies, D. George, T. N. T. Goodman, S. Jaffard, C. Li, S. Mallat, Y. Meyer, C. A. Micchelli, E. Quak, X. L. Shi, J. Stöckler, J. Z. Wang, J. D. Ward, and R. Wells. Among my friends who have read portions of the manuscript and made many valuable suggestions, I am especially indebted to C. Li, E. Quak, X. L. Shi, and N. Sivakumar. As usual, I have again enjoyed superb assistance from Robin Campbell, who TEXed the entire manuscript, and from Stephanie Sellers and my wife, Margaret, who produced the manuscript in camera-ready form. Finally, to the editorial office of Academic Press, and particularly to Charles Glaser, who has complete confidence in me, I wish to express my appreciation of their efficient assistance and friendly cooperation.

College Station, Texas | Charles K. Chui
October, 1991

Preface to the second printing

The second printing gave me an opportunity to make some corrections and append two tables of weights for implementing spline-wavelet reconstruction and decomposition. The inclusion of these numerical values was suggested by David Donoho to whom I am very grateful. I would also like to thank my student Jun Zha for his assistance in producing these two tables.

April, 1992 | C. K. C.

1 An Overview

"Wavelets" has been a very popular topic of conversations in many scientific and engineering gatherings these days. Some view wavelets as a new basis for representing functions, some consider it as a technique for time-frequency analysis, and others think of it as a new mathematical subject. Of course, all of them are right, since "wavelets" is a versatile tool with very rich mathematical content and great potential for applications. However, as this subject is still in the midst of rapid development, it is definitely too early to give a unified presentation. The objective of this book is very modest: it is intended to be used as a textbook for an introductory one-semester course on "wavelet analysis" for upper-division undergraduate or beginning graduate mathematics and engineering students, and is also written for both mathematicians and engineers who wish to learn about the subject. For the specialists, this volume is suitable as complementary reading to the more advanced monographs, such as the two volumes of *Ondelettes et Opérateurs* by Yves Meyer, the edited volume of *Wavelets–A Tutorial in Theory and Applications* in this series, and the forthcoming CBMS volume by Ingrid Daubechies.

Since wavelet analysis is a relatively new subject and the approach and organization in this book are somewhat different from that in the others, the goal of this chapter is to convey a general idea of what wavelet analysis is about and to describe what this book aims to cover.

1.1. From Fourier analysis to wavelet analysis

Let $L^2(0,2\pi)$ denote the collection of all measurable functions f defined on the interval $(0,2\pi)$ with

$$\int_0^{2\pi} |f(x)|^2 dx < \infty.$$

For the reader who is not familiar with the basic Lebesgue theory, the sacrifice is very minimal by assuming that f is a piecewise continuous function. It will always be assumed that functions in $L^2(0,2\pi)$ are extended periodically to the real line

$$\mathbb{R} := (-\infty, \infty),$$

namely: $f(x) = f(x-2\pi)$ for all x. Hence, the collection $L^2(0,2\pi)$ is often called the space of 2π-periodic square-integrable functions. That $L^2(0,2\pi)$ is a vector space can be verified very easily. Any f in $L^2(0, 2\pi)$ has a Fourier series representation:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{1.1.1}$$

where the constants c_n, called the Fourier coefficients of f, are defined by

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx. \tag{1.1.2}$$

The convergence of the series in (1.1.1) is in $L^2(0, 2\pi)$, meaning that

$$\lim_{M,N\to\infty} \int_0^{2\pi} \left| f(x) - \sum_{n=-M}^{N} c_n e^{inx} \right|^2 dx = 0.$$

There are two distinct features in the Fourier series representation (1.1.1). First, we mention that f is decomposed into a sum of infinitely many mutually orthogonal components $g_n(x) := c_n e^{inx}$, where orthogonality means that

$$\langle g_m, g_n \rangle^* = 0, \quad \text{for all} \quad m \neq n, \tag{1.1.3}$$

with the "*inner product*" in (1.1.3) being defined by

$$\langle g_m, g_n \rangle^* := \frac{1}{2\pi} \int_0^{2\pi} g_m(x) \overline{g_n(x)} dx. \tag{1.1.4}$$

That (1.1.3) holds is a consequence of the important, yet simple fact that

$$w_n(x) := e^{inx}, \qquad n = \cdots, -1, 0, 1, \ldots, \tag{1.1.5}$$

is an orthonormal (o.n.) basis of $L^2(0, 2\pi)$. The second distinct feature of the Fourier series representation (1.1.1) is that the o.n. basis $\{w_n\}$ is generated by "*dilation*" of a single function

$$w(x) := e^{ix}; \tag{1.1.6}$$

that is, $w_n(x) = w(nx)$ for all integers n. This will be called *integral dilation*.

Let us summarize this remarkable fact by saying that *every 2π-periodic square-integrable function is generated by a "superposition" of integral dilations of the basic function $w(x) = e^{ix}$.*

We also remark that from the o.n. property of $\{w_n\}$, the Fourier series representation (1.1.1) also satisfies the so-called *Parseval Identity*:

$$\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2. \tag{1.1.7}$$

Let ℓ^2 denote the space of all square-summable bi-infinite sequences; that is, $\{c_n\} \in \ell^2$ if and only if

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty.$$

Hence, if the square-root of the quantity on the left of (1.1.7) is used as the "norm" for the measurement of functions in $L^2(0, 2\pi)$, and similarly, the square-root of the quantity on the right of (1.1.7) is used as the norm for ℓ^2, then the function space $L^2(0, 2\pi)$ and the sequence space ℓ^2 are "*isometric*" to each other. Returning to the above mentioned observation on the Fourier series representation (1.1.1), we can also say that *every 2π-periodic square-integrable function is an ℓ^2-linear combination of integral dilations of the basic function $w(x) = e^{ix}$.*

We emphasize again that the basic function

$$w(x) = e^{ix} = \cos x + i \sin x,$$

which is a "*sinusoidal wave*", is the *only* function required to generate all 2π-periodic square-summable functions. For any integer n with large absolute value, the wave $w_n(x) = w(nx)$ has high "*frequency*", and for n with small absolute value, the wave w_n has low frequency. So, every function in $L^2(0, 2\pi)$ is composed of waves with various frequencies.

We next consider the space $L^2(\mathbb{R})$ of measurable functions f, defined on the real line $\mathbb{R}$, that satisfy

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

Clearly, the two function spaces $L^2(0, 2\pi)$ and $L^2(\mathbb{R})$ are quite different. In particular, since (the local average values of) every function in $L^2(\mathbb{R})$ must "decay" to zero at $\pm\infty$, the sinusoidal (wave) functions w_n do not belong to $L^2(\mathbb{R})$. In fact, if we look for "waves" that generate $L^2(\mathbb{R})$, these waves should decay to zero at $\pm\infty$; and for all practical purposes, the decay should be very fast. That is, we look for small waves, or "*wavelets*", to generate $L^2(\mathbb{R})$. As in the situation of $L^2(0, 2\pi)$, where one single function $w(x) = e^{ix}$ generates the entire space, we also prefer to have a single function, say ψ, to generate all of $L^2(\mathbb{R})$. But if the wavelet ψ has very fast decay, how can it cover the whole real line? The obvious way is to shift ψ along $\mathbb{R}$.

Let $\mathbb{Z}$ denote the set of integers:

$$\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}.$$

The simplest way for ψ to cover all of $\mathbb{R}$ is to consider all the *integral shifts* of ψ, namely:

$$\psi(x - k), \qquad k \in \mathbb{Z}.$$

Next, as in the sinusoidal situation, we must also consider waves with different frequencies. For various reasons which will soon be clear to the reader, we do not wish to consider "single-frequency" waves, but rather, waves with frequencies partitioned into consecutive "octaves" (or frequency bands). For computational efficiency, we will use integral powers of 2 for frequency partitioning; that is, we now consider the small waves

$$\psi(2^j x - k), \qquad j,k \in \mathbb{Z}. \tag{1.1.8}$$

Observe that $\psi(2^j x - k)$ is obtained from a single "wavelet" function $\psi(x)$ by a *binary dilation* (i.e. dilation by 2^j) and a *dyadic translation* (of $k/2^j$).

So, we are interested in "wavelet" functions ψ whose binary dilations and dyadic translations are enough to represent all the functions in $L^2(\mathbb{R})$. For simplicity, let us first consider an orthogonal basis generated by ψ. Later in this chapter (see Section 1.4), we will introduce the more general "wavelet series".

Throughout this book, we will use the following notations for the *inner product* and *norm* for the space $L^2(\mathbb{R})$:

$$\langle f,g\rangle := \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx; \tag{1.1.9}$$

$$\|f\|_2 := \langle f,f\rangle^{1/2}, \tag{1.1.10}$$

where $f,g \in L^2(\mathbb{R})$. Note that for any $j,k \in \mathbb{Z}$, we have

$$\begin{aligned}\|f(2^j\cdot - k)\|_2 &= \left\{\int_{-\infty}^{\infty} |f(2^j x - k)|^2 dx\right\}^{1/2}\\ &= 2^{-j/2}\|f\|_2.\end{aligned}$$

Hence, if a function $\psi \in L^2(\mathbb{R})$ has unit length, then all of the functions $\psi_{j,k}$, defined by

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \qquad j,k \in \mathbb{Z}, \tag{1.1.11}$$

also have unit length; that is,

$$\|\psi_{j,k}\|_2 = \|\psi\|_2 = 1, \qquad j,k \in \mathbb{Z}. \tag{1.1.12}$$

In this book, the Kronecker symbol

$$\delta_{j,k} := \begin{cases} 1 & \text{for } \; j = k; \\ 0 & \text{for } \; j \neq k, \end{cases} \tag{1.1.13}$$

defined on $\mathbb{Z} \times \mathbb{Z}$, will be often used.

Definition 1.1. *A function $\psi \in L^2(\mathbb{R})$ is called an orthogonal wavelet (or o.n. wavelet), if the family $\{\psi_{j,k}\}$, as defined in (1.1.11), is an orthonormal basis of $L^2(\mathbb{R})$; that is,*

$$\langle \psi_{j,k}, \psi_{\ell,m}\rangle = \delta_{j,\ell}\cdot\delta_{k,m}, \qquad j,k,\ell,m \in \mathbb{Z}, \tag{1.1.14}$$

and every $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{j,k}\psi_{j,k}(x), \tag{1.1.15}$$

where the convergence of the series in (1.1.15) is in $L^2(\mathbb{R})$, namely:

$$\lim_{M_1,N_1,M_2,N_2\to\infty} \left\| f - \sum_{j=-M_2}^{N_2} \sum_{k=-M_1}^{N_1} c_{j,k}\psi_{j,k} \right\|_2 = 0.$$

The simplest example of an orthogonal wavelet is the Haar function ψ_H defined by

$$\psi_H(x) := \begin{cases} 1 & \text{for} \quad 0 \le x < \frac{1}{2}; \\ -1 & \text{for} \quad \frac{1}{2} \le x < 1; \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.16}$$

We will give a brief discussion of this function in Sections 1.5 and 1.6. Other o.n. wavelets will be studied in some details in Chapter 7.

The series representation of f in (1.1.15) is called a *wavelet series*. Analogous to the notion of Fourier coefficients in (1.1.2), the wavelet coefficients $c_{j,k}$ are given by

$$c_{j,k} = \langle f, \psi_{j,k} \rangle. \tag{1.1.17}$$

That is, if we define an integral transform W_ψ on $L^2(\mathbb{R})$ by

$$(W_\psi f)(b,a) := |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x)\overline{\psi(\frac{x-b}{a})}dx, \qquad f \in L^2(\mathbb{R}), \tag{1.1.18}$$

then the wavelet coefficients in (1.1.15) and (1.1.17) become

$$c_{j,k} = (W_\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right). \tag{1.1.19}$$

The linear transformation W_ψ is called the "*integral wavelet transform*" relative to the "basic wavelet" ψ. Hence, the $(j,k)^{\text{th}}$ *wavelet coefficient of f is given by the integral wavelet transformation of f evaluated at the dyadic position $b = k/2^j$ with binary dilation $a = 2^{-j}$*, where the same o.n. wavelet ψ is used to generate the wavelet series (1.1.15) and to define the integral wavelet transform (1.1.18).

The importance of the integral wavelet transform will be discussed in the next section. Here, we only mention that this integral transform greatly enhances the value of the (integral) Fourier transform $\mathcal{F}$, defined by

$$(\mathcal{F}f)(y) := \int_{-\infty}^{\infty} e^{-iyx} f(x)dx, \qquad f \in L^2(\mathbb{R}). \tag{1.1.20}$$

The mathematical treatment of this transform will be delayed to the next chapter. As is well known, the Fourier transform is the other important component of Fourier analysis. Hence, it is interesting to note that while the two components of Fourier analysis, namely: the Fourier series and the Fourier transform, are basically unrelated; the two corresponding components of wavelet analysis, namely: the wavelet series (1.1.15) and the integral wavelet transform (1.1.18), have an intimate relationship as described by (1.1.19).

1.2. The integral wavelet transform and time-frequency analysis

The Fourier transform $\mathcal{F}$ defined in (1.1.20) not only is a very powerful mathematical tool, but also has very significant physical interpretations in applications. For instance, if a function $f \in L^2(\mathbb{R})$ is considered as an *analog signal* with *finite energy*, defined by its norm $\|f\|_2$, then the Fourier transform

$$\hat{f}(\omega) := (\mathcal{F}f)(\omega) \tag{1.2.1}$$

of f represents the *spectrum* of this signal. In signal analysis, analog signals are defined in the *time-domain*, and the spectral information of these signals is given in the *frequency-domain*. To facilitate our presentation, we will allow negative frequencies for the time being. Hence, both the time- and frequency-domains are the real line $\mathbb{R}$. Analogous to the Parseval Identity for Fourier series, the Parseval Identity that describes the relationship between functions in $L^2(\mathbb{R})$ and their Fourier transforms is given by

$$\langle f, g\rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{g}\rangle, \qquad f, g \in L^2(\mathbb{R}). \tag{1.2.2}$$

Here, the notation of inner product introduced in (1.1.9) is used, and as will be seen in the next chapter, the Fourier transformation $\mathcal{F}$ takes $L^2(\mathbb{R})$ onto itself. As a consequence of (1.2.2), we observe that the energy of an analog signal is directly proportional to its spectral content; more precisely,

$$\|f\|_2 = \frac{1}{\sqrt{2\pi}}\|\hat{f}\|_2, \qquad f \in L^2(\mathbb{R}). \tag{1.2.3}$$

However, the formula

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-it\omega} f(t)dt \tag{1.2.4}$$

of the Fourier transform alone is quite inadequate for most applications. In the first place, to extract the spectral information $\hat{f}(\omega)$ from the analog signal $f(t)$ from this formula, it takes an infinite amount of time, using both past and future information of the signal just to evaluate the spectrum at a single frequency ω. Besides, the formula (1.2.4) does not even reflect frequencies that evolve with time. What is really needed is for one to be able to determine the time intervals that yield the spectral information on any desirable range of frequencies (or frequency band). In addition, since the frequency of a signal is directly proportional to the length of its cycle, it follows that for high-frequency spectral information, the time-interval should be relatively small to give better accuracy, and for low-frequency spectral information, the time-interval should be relatively wide to give complete information. In other words, it is important to have a flexible time-frequency window that automatically narrows at high "*center-frequency*" and widens at low center-frequency. Fortunately, the integral wavelet transform W_ψ relative to some "basic wavelet" ψ, as introduced in (1.1.18), has this so-called zoom-in and zoom-out capability.

To be more specific, both ψ and its Fourier transform $\widehat{\psi}$ must have sufficiently fast decay so that they can be used as "*window functions*". For an $L^2(\mathbb{R})$ function w to qualify as a window function, it must be possible to identify its "center" and "width", which are defined as follows.

Definition 1.2. *A nontrivial function $w \in L^2(\mathbb{R})$ is called a window function if $xw(x)$ is also in $L^2(\mathbb{R})$. The center t^* and radius Δ_w of a window function w are defined to be*

$$t^* := \frac{1}{\|w\|_2^2} \int_{-\infty}^{\infty} x|w(x)|^2 dx \tag{1.2.5}$$

and

$$\Delta_w := \frac{1}{\|w\|_2} \left\{ \int_{-\infty}^{\infty} (x - t^*)^2 |w(x)|^2 dx \right\}^{1/2}, \tag{1.2.6}$$

respectively; and the width of the window function w is defined by $2\Delta_w$.

We have not formally defined a "basic wavelet" ψ yet and will not do so until the next section. An example of a basic wavelet is any orthogonal wavelet as already discussed in the previous section. In any case, we will see that any basic wavelet window function must necessarily satisfy:

$$\int_{-\infty}^{\infty} \psi(x) dx = 0, \tag{1.2.7}$$

so that its graph is a *small wave*.

Suppose that ψ is any basic wavelet such that both ψ and its Fourier transform $\widehat{\psi}$ are window functions with centers and radii given by $t^*, \omega^*, \Delta_\psi, \Delta_{\widehat{\psi}}$, respectively. Then in the first place, it is clear that the integral wavelet transform

$$(W_\psi f)(b, a) = |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt \tag{1.2.8}$$

of an analog signal f, as introduced in (1.1.18), localizes the signal with a "time window"

$$[b + at^* - a\Delta_\psi,\ b + at^* + a\Delta_\psi],$$

where the center of the window is at $b + at^*$ and the width is given by $2a\Delta_\psi$. This is called "time-localization" in signal analysis. On the other hand, if we set

$$\eta(\omega) := \widehat{\psi}(\omega + \omega^*), \tag{1.2.9}$$

then η is also a window function with center at 0 and radius given by $\Delta_{\widehat{\psi}}$; and by the Parseval Identity (1.2.2), the integral wavelet transform in (1.2.8) becomes

$$(W_\psi f)(b,a) = \frac{a|a|^{-\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{ib\omega}\overline{\eta\left(a\left(\omega - \frac{\omega^*}{a}\right)\right)}d\omega. \tag{1.2.10}$$

Hence, with the exception of multiplication by $a|a|^{-\frac{1}{2}}/2\pi$ and a linear phase-shift of $e^{ib\omega}$, determined by the amount of translation of the time-window, the same quantity $(W_\psi f)(b,a)$ also gives localized information of the spectrum $\hat{f}(\omega)$ of the signal $f(t)$, with a "frequency window"

$$\left[\frac{\omega^*}{a} - \frac{1}{a}\Delta_{\widehat{\psi}}, \frac{\omega^*}{a} + \frac{1}{a}\Delta_{\widehat{\psi}}\right],$$

whose center is at ω^*/a and whose width is given by $2\Delta_{\widehat{\psi}}/a$. This is called "frequency-localization". By equating the quantities (1.2.8) and (1.2.10), we now have a "time-frequency window":

$$[b + at^* - a\Delta_\psi,\ b + at^* + a\Delta_\psi] \times \left[\frac{\omega^*}{a} - \frac{1}{a}\Delta_{\widehat{\psi}}, \frac{\omega^*}{a} + \frac{1}{a}\Delta_{\widehat{\psi}}\right] \tag{1.2.11}$$

for time-frequency analysis using the integral wavelet transform relative to a basic wavelet ψ with the window conditions described above.

Several comments are in order. First, since we must eventually consider positive frequencies, the basic wavelet ψ should be so chosen that the center ω^* of $\widehat{\psi}$ is a positive number. In practice, this positive number, along with the positive scaling parameter a, is selected in such a way that ω^*/a is the "*center-frequency*" of the "*frequency band*" $[\frac{\omega^*}{a} - \frac{1}{a}\Delta_{\widehat{\psi}}, \frac{\omega^*}{a} + \frac{1}{a}\Delta_{\widehat{\psi}}]$ of interest. Then the ratio of the center-frequency to the width of the frequency band is given by

$$\frac{\omega^*/a}{2\Delta_{\widehat{\psi}}/a} = \frac{\omega^*}{2\Delta_{\widehat{\psi}}}, \tag{1.2.12}$$

which is independent of the location of the center-frequency. This is called "*constant-Q*" frequency analysis. The importance of the time-frequency window (1.2.11) is that it narrows for large center-frequency ω^*/a and widens for small center-frequency ω^*/a (cf. Figure 1.2.1), although the area of the window is a constant, given by $4\Delta_\psi\Delta_{\widehat{\psi}}$. This is exactly what is most desirable in time-frequency analysis. Details will be studied in Chapter 3.